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Gronwall–Bellman type nonlinear delay integral inequalities on time scales

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ABSTRACT

In this paper, some Gronwall–Bellman type nonlinear delay integral inequalities on time scales are established, which provide a handy tool in deriving boundedness of solutions of certain delay dynamic equations on time scales. Our results generalize some of the main results in Lipovan (2006) [1], Pachpatte (2000) [2], Ferreira and Torres (2009) [3], Zhang and Meng (2008) [4], Cheung and Ren (2006) [5], Kim (2009) [6], and some of our results unify continuous and discrete analysis in the literature.

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1. Introduction

During the past decades, with the development of the theory of differential and integral equations, a lot of integral and difference inequalities, for example, [1–16] and the references therein, have been discovered, which play an important role in the research of boundedness, global existence, stability of solutions of differential and integral equations as well as difference equations.

In these inequalities, Gronwall–Bellman type inequalities and their generations have been paid much attention by many authors (for example, see [1–6]). The generations of Gronwall–Bellman inequalities mentioned above have proven to be very effective in the study of qualitative and quantitative properties of solutions of differential and integral equations as well as certain difference equations.

In this paper, we extend some of the presented inequalities in [1–6] to delay inequalities on arbitrary time scales, and new bounds for unknown functions are obtained due to the presented inequalities.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, while $\mathbb{Z}$ denotes the set of integers. For two given sets G, H, we denote the set of maps from G to H by (G, H).

2. Some preliminaries on time scales

The development of the theory of time scales was initiated by Hilger [17] in 1988 as a theory capable to contain both difference and differential calculus in a consistent way. Since then many authors have expounded on various aspects of the theory of dynamic equations on time scales, for example [18–20], and the references therein. However, Gronwall–Bellman type delay integral inequalities on time scales have been payed little attention. Recent results in this direction include the works of Li [21], and Ma [22].

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A time scale is an arbitrary nonempty closed subset of the real numbers. In this paper, $\mathbb{T}$ denotes an arbitrary time scale. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma(t) \in (\mathbb{T}, \mathbb{T})$ and $\rho(t) \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T},\ s < t\}$.

Definition 2.1. The graininess $\mu \in (\mathbb{T}, \mathbb{R}_+)$ is defined by $\mu(t) = \sigma(t) - t$.

Remark 2.1. Obviously, $\mu(t) = 0$ if $\mathbb{T} = \mathbb{R}$ while $\mu(t) = 1$ if $\mathbb{T} = \mathbb{Z}$.

Definition 2.2. A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$.

Definition 2.3. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum, otherwise it is $\mathbb{T}$ without the left-scattered maximum.

Definition 2.4. A function $f \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while f is called regressive if $1 + \mu(t) f(t) \neq 0$. C_{rd} denotes the set of rd-continuous functions, while $\mathfrak{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathfrak{R}^+ = \{f \mid f \in \mathfrak{R},\ 1 + \mu(t) f(t) > 0, \forall t \in \mathbb{T}\}$.

Definition 2.5. For some $t \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T}, \mathbb{R})$, the *delta derivative* of f is denoted by $f^\Delta(t)$, and satisfies

$$\left|f\big(\sigma(t)\big) - f(s) - f^\Delta(t)\big(\sigma(t) - s\big)\right| \leqslant \varepsilon\left|\sigma(t) - s\right| \quad \text{for } \forall \varepsilon > 0,$$

where $s \in \mathfrak{U}$, and $\mathfrak{U}$ is a neighborhood of t. The function f is called *delta differential* on $\mathbb{T}^\kappa$.

Similarly, for some $x \in \mathbb{T}^\kappa$, and a function $f \in (\mathbb{T} \times \mathbb{T}, \mathbb{R})$, the *partial delta derivative* of f with respect to x is denoted by $f_x^\Delta(x, y)$, and satisfies

$$\left|f\big(\sigma(x), y\big) - f(s, y) - f_x^\Delta(x, y)\big(\sigma(x) - s\big)\right| \leqslant \varepsilon\left|\sigma(x) - s\right| \quad \text{for } \forall \varepsilon > 0,$$

where $s \in \mathfrak{U}$, and $\mathfrak{U}$ is a neighborhood of x.

Remark 2.2. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t)$ becomes the usual derivative $f'(t)$, while $f^\Delta(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$, which represents the forward difference.

Definition 2.6. For $a, b \in \mathbb{T}$ and a function $f \in (\mathbb{T}, \mathbb{R})$, the *Cauchy integral* of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \text{where } F^\Delta(t) = f(t),\ t \in \mathbb{T}^\kappa.$$

Similarly, for $a, b \in \mathbb{T}$ and a function $f \in (\mathbb{T} \times \mathbb{T}, \mathbb{R})$, the *Cauchy partial integral* of f with respect to x is defined by

$$\int_a^b f(x, y)\Delta x = F(b, y) - F(a, y), \quad \text{where } F_x^\Delta(x, y) = f(x, y),\ x \in \mathbb{T}^\kappa.$$

For more details about the calculus of time scales, we advise the reader to refer to [23].

3. Main results

Lemma 3.1. *(See [24, Theorem 2.2].) Let $t_0 \in \mathbb{T}^\kappa$ and $\omega : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ be continuous at (t, t), where $t \geqslant t_0$, $t \in \mathbb{T}^\kappa$ with $t > t_0$. Assume that $\omega^\Delta(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. If for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$\left|\omega\big(\sigma(t), \tau\big) - \omega(s, t) - \omega^\Delta(t, \tau)\big(\sigma(t) - s\big)\right| \leqslant \varepsilon\left|\sigma(t) - s\right| \quad \textit{for all } s \in U,$$

where ω^Δ denotes the derivative of ω with respect to the first variable, then

$$g(t) := \int_{t_0}^t \omega(t, \tau)\Delta\tau$$

implies

$$g^{\Delta}(t) = \int_{t_0}^{t} \omega^{\Delta}(t,\tau)\Delta\tau + \omega\big(\sigma(t),t\big).$$

3.1. 1D cases

For convenience of notation, we denote $\mathbb{T}_0 = [t_0,\infty) \cap \mathbb{T}$, where $t_0 \in \mathbb{T}^{\kappa}$, and furthermore assume $\mathbb{T}_0 \subseteq \mathbb{T}^{\kappa}$.

Theorem 3.1. *Suppose $u(t), a(t) \in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$, and a is nondecreasing. $f(t,s)$, $f_t^{\Delta}(t,s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0,\mathbb{R}_+)$. $\omega \in C(\mathbb{R}_+,\mathbb{R}_+)$ is nondecreasing. $p > 0$ is a constant. $\tau \in (\mathbb{T}_0,\mathbb{T})$, $\tau(t) \leqslant t$, $-\infty < \alpha = \inf\{\tau(t),\ t \in \mathbb{T}_0\} \leqslant t_0$. $\phi \in C_{rd}([\alpha,t_0] \cap \mathbb{T},\mathbb{R}_+)$. If for $t \in \mathbb{T}_0$, $u(t)$ satisfies the following inequality*

$$u^p(t) \leqslant a(t) + \int_{t_0}^{t} f(t,s)\omega\big(u\big(\tau(s)\big)\big)\Delta s, \tag{3.1}$$

with the initial condition

$$\begin{cases} u(t) = \phi(t) & \text{if } t \in [\alpha,t_0] \cap \mathbb{T}, \\ \phi\big(\tau(t)\big) \leqslant a^{\frac{1}{p}}(t) & \text{if } \tau(t) \leqslant t_0,\ \forall t \in \mathbb{T}_0, \end{cases} \tag{3.2}$$

then

$$u(t) \leqslant \left\{G^{-1}\left[G\big(a(t)\big) + \int_{t_0}^{t} f(t,s)\Delta s\right]\right\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0, \tag{3.3}$$

where G is an increasing bijective function, and

$$\big[G\big(z(t)\big)\big]^{\Delta} = \frac{z^{\Delta}(t)}{\omega(z^{\frac{1}{p}}(t))}. \tag{3.4}$$

Proof. Let $T \in \mathbb{T}_0$ be fixed, and $t \in [t_0,T] \cap \mathbb{T}$. Denote

$$v(t) = a(T) + \int_{t_0}^{t} f(t,s)\omega\big(u\big(\tau(s)\big)\big)\Delta s.$$

Then for $t \in [t_0,T] \cap \mathbb{T}$, we have

$$u(t) \leqslant v^{\frac{1}{p}}(t). \tag{3.5}$$

Furthermore, for $t \in [t_0,T] \cap \mathbb{T}$, if $\tau(t) \geqslant t_0$, then $\tau(t) \in [t_0,T] \cap \mathbb{T}$, and from (3.5) we have

$$u\big(\tau(t)\big) \leqslant v^{\frac{1}{p}}\big(\tau(t)\big) \leqslant v^{\frac{1}{p}}(t) \tag{3.6}$$

if $\tau(t) \leqslant t_0$, from (3.2) we obtain

$$u\big(\tau(t)\big) = \phi\big(\tau(t)\big) \leqslant a^{\frac{1}{p}}(t) \leqslant v^{\frac{1}{p}}(t). \tag{3.7}$$

So from (3.6) and (3.7) we always have

$$u\big(\tau(t)\big) \leqslant v^{\frac{1}{p}}(t). \tag{3.8}$$

Moreover, by Lemma 3.1 we have

$$\begin{aligned} v^{\Delta}(t) &= \int_{t_0}^{t} f_t^{\Delta}(t,s)\omega\big(u\big(\tau(s)\big)\big)\Delta s + f\big(\sigma(t),t\big)\omega\big(u\big(\tau(t)\big)\big) \leqslant \int_{t_0}^{t} f_t^{\Delta}(t,s)\omega\big(v^{\frac{1}{p}}(s)\big)\Delta s + f\big(\sigma(t),t\big)\omega\big(v^{\frac{1}{p}}(t)\big) \\ &\leqslant \left[\int_{t_0}^{t} f_t^{\Delta}(t,s)\Delta s + f\big(\sigma(t),t\big)\right]\omega\big(v^{\frac{1}{p}}(t)\big) = \left[\int_{t_0}^{t} f(t,s)\Delta s\right]^{\Delta}\omega\big(v^{\frac{1}{p}}(t)\big), \end{aligned} \tag{3.9}$$

that is,

$$\frac{v^{\Delta}(t)}{\omega(v^{\frac{1}{p}}(t))} \leqslant \left[\int_{t_0}^{t} f(t,s)\Delta s\right]^{\Delta}. \tag{3.10}$$

An integration for (3.10) with respect to t from t_0 to t yields

$$G\big(v(t)\big) - G\big(v(t_0)\big) \leqslant \int_{t_0}^{t} f(t,s)\Delta s, \tag{3.11}$$

where G is defined in (3.4).

Considering $v(t_0) = a(T)$, and G is increasing, then (3.11) implies

$$v(t) \leqslant G^{-1}\left[G\big(a(T)\big) + \int_{t_0}^{t} f(t,s)\Delta s\right], \quad t \in [t_0, T] \cap \mathbb{T}. \tag{3.12}$$

Combining (3.12), (3.5) we obtain

$$u(t) \leqslant \left\{G^{-1}\left[G\big(a(T)\big) + \int_{t_0}^{t} f(t,s)\Delta s\right]\right\}^{\frac{1}{p}}, \quad t \in [t_0, T] \cap \mathbb{T}. \tag{3.13}$$

Take $t = T$ in (3.13), we obtain

$$u(T) \leqslant \left\{G^{-1}\left[G\big(a(T)\big) + \int_{t_0}^{T} f(T,s)\Delta s\right]\right\}^{\frac{1}{p}}. \tag{3.14}$$

Considering $T \in \mathbb{T}_0$ is arbitrary, then after substituting T with t in (3.14) we obtain the desired result. □

In Theorem 3.1, if we change the conditions for a, b, ω, then we can obtain another bound for the function $u(t)$.

Theorem 3.2. *Suppose u, f, p, ϕ, α, τ are the same as in Theorem* 3.1, *$a, b \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$, and a, b are not necessarily nondecreasing, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing, subadditive, and submultiplicative, that is, for $\forall\alpha \geqslant 0$, $\beta \geqslant 0$ we always have $\omega(\alpha+\beta) \leqslant \omega(\alpha) + \omega(\beta)$ and $\omega(\alpha\beta) \leqslant \omega(\alpha)\omega(\beta)$. If for $t \in \mathbb{T}_0$, $u(t)$ satisfies the following inequality*

$$u^p(t) \leqslant a(t) + b(t)\int_{t_0}^{t} f(s)\omega\big(u\big(\tau(s)\big)\big)\Delta s,$$

with the initial condition (3.2)*, then*

$$u(t) \leqslant \left\{a(t) + b(t)H^{-1}\left[H\big(A(t)\big) + \int_{t_0}^{t} f(s)\omega\left(\frac{1}{p}b(s)\right)\Delta s\right]\right\}^{\frac{1}{p}}, \quad t \in \mathbb{T}_0,$$

where H is an increasing bijective function, and

$$\begin{cases} \big(H\big(v(t)\big)\big)^{\Delta} = \dfrac{v^{\Delta}(t)}{\omega(v(t))}, \\ A(t) = \displaystyle\int_{t_0}^{t} f(s)\omega\left(\frac{1}{p}a(s) + \frac{p-1}{p}\right). \end{cases}$$

The proof for Theorem 3.2 is similar to Theorem 3.1, and we omit it here.

Remark 3.1. If we take $\mathbb{T} = \mathbb{R}$, $p = 1$, $t_0 = 0$, then Theorem 3.1 reduces to [1, Theorem 2.1]. If furthermore $f(t,s) = f_1(t)f_2(s)$, then Theorem 3.1 reduces to [1, Theorem 2.2].

Remark 3.2. Theorem 3.2 unifies some known continuous and discrete analysis in the literature. If we take $\mathbb{T}=\mathbb{R}$, $t_0=0$, $\tau(t)=t$, then Theorem 3.2 reduces to [2, Theorem 2(b3)]. If we take $\mathbb{T}=\mathbb{Z}$, $t_0=0$, $\tau(t)=t$, then Theorem 3.2 reduces to [2, Theorem 4(d3)].

Theorem 3.3. *Suppose u, a, f, ω, τ, α, ϕ are the same as in Theorem 3.1. $g(t,s), g_t^{\Delta}(t,s)\in C_{rd}(\mathbb{T}_0\times\mathbb{T}_0,\mathbb{R}_+)$. $h\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$. $\psi,\eta\in C(\mathbb{R}_+,\mathbb{R}_+)$, and ψ is nondecreasing, while η is strictly increasing. If for $t\in\mathbb{T}_0$, $u(t)$ satisfies the following inequality*

$$\eta\big(u(t)\big)\leqslant a(t)+\int_{t_0}^{t}\big[f(t,s)\psi\big(u\big(\tau(s)\big)\big)\omega\big(u\big(\tau(s)\big)\big)+g(t,s)\psi\big(u\big(\tau(s)\big)\big)\big]\Delta s+\int_{t_0}^{t}\int_{t_0}^{s}h(\xi)\psi\big(u\big(\tau(\xi)\big)\big)\Delta\xi\,\Delta s, \tag{3.15}$$

with the initial condition

$$\begin{cases}\eta\big(u(t)\big)=\phi(t) & \text{if } t\in[\alpha,t_0]\cap\mathbb{T},\\ \phi\big(\tau(t)\big)\leqslant a(t) & \text{if } \tau(t)\leqslant t_0,\ \forall t\in\mathbb{T}_0,\end{cases}$$

then

$$u(t)\leqslant\eta^{-1}\left\{\widetilde{G}^{-1}\left\{\widetilde{H}^{-1}\left\{\widetilde{H}\left\{\widetilde{G}\big(a(t)\big)+\int_{t_0}^{t}\left[g(t,s)+\int_{t_0}^{s}h(\xi)\Delta\xi\right]\Delta s\right\}+\int_{t_0}^{t}f(t,s)\Delta s\right\}\right\}\right\},\quad t\in\mathbb{T}_0, \tag{3.16}$$

where $\widetilde{G}$, $\widetilde{H}$ are increasing bijective functions, and

$$\begin{cases}\big[\widetilde{G}\big(v(t)\big)\big]^{\Delta}=\dfrac{v^{\Delta}(t)}{\psi(\eta^{-1}(v(t)))},\\[2ex] \big[\widetilde{H}\big(z(t)\big)\big]^{\Delta}=\dfrac{z^{\Delta}(t)}{\omega(\eta^{-1}(\widetilde{G}^{-1}(z(t))))}.\end{cases} \tag{3.17}$$

Proof. Let $T\in\mathbb{T}_0$ be fixed and $t\in[t_0,T]\cap\mathbb{T}$. Denote

$$v(t)=a(T)+\int_{t_0}^{t}\big[f(t,s)\psi\big(u\big(\tau(s)\big)\big)\omega\big(u\big(\tau(s)\big)\big)+g(t,s)\psi\big(u\big(\tau(s)\big)\big)\big]\Delta s+\int_{t_0}^{t}\int_{t_0}^{s}h(\xi)\psi\big(u\big(\tau(\xi)\big)\big)\Delta\xi\,\Delta s.$$

Then we have

$$u(t)\leqslant\eta^{-1}\big(v(t)\big),\quad t\in[t_0,T]\cap\mathbb{T}. \tag{3.18}$$

Similar to the process of (3.6)–(3.8) we obtain

$$u\big(\tau(t)\big)\leqslant\eta^{-1}\big(v(t)\big). \tag{3.19}$$

Furthermore, by Lemma 3.1 we have

$$\begin{aligned}v^{\Delta}(t)={}&\int_{t_0}^{t}\big[f_t^{\Delta}(t,s)\psi\big(u\big(\tau(s)\big)\big)\omega\big(u\big(\tau(s)\big)\big)+g_t^{\Delta}(t,s)\psi\big(u\big(\tau(s)\big)\big)\big]\Delta s+f\big(\sigma(t),t\big)\psi\big(u\big(\tau(s)\big)\big)\omega\big(u\big(\tau(s)\big)\big)\\&+g\big(\sigma(t),t\big)\psi\big(u\big(\tau(s)\big)\big)+\int_{t_0}^{t}h(\xi)\psi\big(u\big(\tau(\xi)\big)\big)\Delta\xi\\ \leqslant{}&\left\{\int_{t_0}^{t}\big[f_t^{\Delta}(t,s)\omega\big(\eta^{-1}\big(v(s)\big)\big)+g_t^{\Delta}(t,s)\big]\Delta s+f\big(\sigma(t),t\big)\omega\big(\eta^{-1}\big(v(s)\big)\big)+g\big(\sigma(t),t\big)+\int_{t_0}^{t}h(\xi)\Delta\xi\right\}\\&\times\psi\big(\eta^{-1}\big(v(t)\big)\big),\end{aligned}$$

which implies

$$\begin{aligned}\frac{v^{\Delta}(t)}{\psi(\eta^{-1}(v(t)))}&\leqslant\int_{t_0}^{t}\big[f_t^{\Delta}(t,s)\omega\big(\eta^{-1}\big(v(s)\big)\big)+g_t^{\Delta}(t,s)\big]\Delta s+f\big(\sigma(t),t\big)\omega\big(\eta^{-1}\big(v(s)\big)\big)+g\big(\sigma(t),t\big)+\int_{t_0}^{t}h(\xi)\Delta\xi\\&=\left\{\int_{t_0}^{t}\left[f(t,s)\omega\big(\eta^{-1}\big(v(s)\big)\big)+g(t,s)+\int_{t_0}^{s}h(\xi)\Delta\xi\right]\Delta s\right\}^{\Delta}.\end{aligned} \tag{3.20}$$

An integration for (3.20) with respect to t from t_0 to t yields

$$\widetilde{G}\big(v(t)\big)-\widetilde{G}\big(v(t_0)\big)\leqslant\int_{t_0}^{t}\left[f(t,s)\omega\big(\eta^{-1}\big(v(s)\big)\big)+g(t,s)+\int_{t_0}^{s}h(\xi)\Delta\xi\right]\Delta s,\tag{3.21}$$

where $\widetilde{G}$ is defined in (3.17).

Considering $v(t_0)=a(T)$, and $\widetilde{G}$ is increasing, then (3.21) implies

$$v(t)\leqslant\widetilde{G}^{-1}\left\{\widetilde{G}\big(a(T)\big)+\int_{t_0}^{t}\left[f(t,s)\omega\big(\eta^{-1}\big(v(s)\big)\big)+g(t,s)+\int_{t_0}^{s}h(\xi)\Delta\xi\right]\Delta s\right\}.\tag{3.22}$$

Let $z(t)=\widetilde{G}(a(T))+\int_{t_0}^{T}[g(T,s)+\int_{t_0}^{s}h(\xi)\Delta\xi]\Delta s+\int_{t_0}^{t}f(t,s)\omega(\eta^{-1}(v(s)))\Delta s$, then we have

$$v(t)\leqslant\widetilde{G}^{-1}\big(z(t)\big),\quad t\in[t_0,T]\cap\mathbb{T},\tag{3.23}$$

and by Lemma 3.1 we have

$$\begin{aligned}z^{\Delta}(t)&=\int_{t_0}^{t}f_t^{\Delta}(t,s)\omega\big(\eta^{-1}\big(v(s)\big)\big)\Delta s+f\big(\sigma(t),t\big)\omega\big(\eta^{-1}\big(v(t)\big)\big)\\&\leqslant\left[\int_{t_0}^{t}f_t^{\Delta}(t,s)\Delta s+f\big(\sigma(t),t\big)\right]\omega\big(\eta^{-1}\big(\widetilde{G}^{-1}\big(z(t)\big)\big)\big),\end{aligned}\tag{3.24}$$

which implies

$$\frac{z^{\Delta}(t)}{\omega(\eta^{-1}(\widetilde{G}^{-1}(z(t))))}\leqslant\int_{t_0}^{t}f_t^{\Delta}(t,s)\Delta s+f\big(\sigma(t),t\big)=\left[\int_{t_0}^{t}f(t,s)\Delta s\right]^{\Delta}.\tag{3.25}$$

Integrating (3.25) with respect to t from t_0 to t yields

$$\widetilde{H}\big(z(t)\big)-\widetilde{H}\big(z(t_0)\big)\leqslant\int_{t_0}^{t}f(t,s)\Delta s,$$

which is followed by

$$\begin{aligned}z(t)&\leqslant\widetilde{H}^{-1}\left\{\widetilde{H}\big(z(t_0)\big)+\int_{t_0}^{t}f(t,s)\Delta s\right\}\\&=\widetilde{H}^{-1}\left\{\widetilde{H}\left\{\widetilde{G}\big(a(T)\big)+\int_{t_0}^{T}\left[g(T,s)+\int_{t_0}^{s}h(\xi)\Delta\xi\right]\Delta s\right\}+\int_{t_0}^{t}f(t,s)\Delta s\right\},\quad t\in[t_0,T]\cap\mathbb{T}.\end{aligned}\tag{3.26}$$

Combining (3.18), (3.23), and (3.26) we obtain

$$u(t)\leqslant\eta^{-1}\left\{\widetilde{G}^{-1}\left\{\widetilde{H}^{-1}\left\{\widetilde{H}\left\{\widetilde{G}\big(a(T)\big)+\int_{t_0}^{T}\left[g(T,s)+\int_{t_0}^{s}h(\xi)\Delta\xi\right]\Delta s\right\}+\int_{t_0}^{t}f(t,s)\Delta s\right\}\right\}\right\},\quad t\in[t_0,T]\cap\mathbb{T}.\tag{3.27}$$

Take $t=T$ in (3.27) yields

$$u(T)\leqslant\eta^{-1}\left\{\widetilde{G}^{-1}\left\{\widetilde{H}^{-1}\left\{\widetilde{H}\left\{\widetilde{G}\big(a(T)\big)+\int_{t_0}^{T}\left[g(T,s)+\int_{t_0}^{s}h(\xi)\Delta\xi\right]\Delta s\right\}+\int_{t_0}^{T}f(T,s)\Delta s\right\}\right\}\right\}.\tag{3.28}$$

Since $T\in\mathbb{T}_0$ is arbitrary, then after substituting T with t in (3.28) we obtain the desired result. □

Remark 3.3. If we take $\mathbb{T}=\mathbb{R}$, $h(t)\equiv 0$, then Theorem 3.3 reduces to [3, Theorem 1].

3.2. 2D cases

For the sake of convenience, we denote $\widehat{\mathbb{T}}_0 = [x_0, \infty) \cap \mathbb{T}$, $\widetilde{\mathbb{T}}_0 = [y_0, \infty) \cap \mathbb{T}$, where $x_0, y_0 \in \mathbb{T}^\kappa$, and furthermore assume $\widehat{\mathbb{T}}_0 \subseteq \mathbb{T}^\kappa$, $\widetilde{\mathbb{T}}_0 \subseteq \mathbb{T}^\kappa$.

Theorem 3.4. *Suppose $u, a \in C_{rd}(\widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R}_+)$, and $a(x, y)$ is nondecreasing. $f(x, s, y, t)$, $f_x^\Delta(x, s, y, t) \in C_{rd}(\widehat{\mathbb{T}}_0^2 \times \widetilde{\mathbb{T}}_0^2, \mathbb{R}_+)$, and $[\int_{y_0}^y f(x, s, y, t)\Delta t]_x^\Delta \geqslant 0$. $p > 0$ is a constant. ω is the same as in Theorem 3.1. $\tau_1 \in (\widehat{\mathbb{T}}_0, \mathbb{T})$, $\tau_1(x) \leqslant x$, $-\infty < \alpha = \inf\{\tau_1(x), x \in \widehat{\mathbb{T}}_0\} \leqslant x_0$. $\tau_2 \in (\widetilde{\mathbb{T}}_0, \mathbb{T})$, $\tau_2(y) \leqslant y$, $-\infty < \beta = \inf\{\tau_2(y), y \in \widetilde{\mathbb{T}}_0\} \leqslant y_0$. $\phi \in C_{rd}(([\alpha, x_0] \times [\beta, y_0]) \cap \mathbb{T}^2, \mathbb{R}_+)$. If for $(x, y) \in \widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0$, $u(x, y)$ satisfies the following inequality*

$$u^p(x, y) \leqslant a(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} f(x, s, y, t)\omega\big(u\big(\tau_1(s), \tau_2(t)\big)\big)\Delta t\Delta s, \tag{3.29}$$

with the initial condition

$$\begin{cases} u(x, y) = \phi(x, y) & \text{if } x \in [\alpha, x_0] \cap \mathbb{T} \text{ or } y \in [\beta, y_0] \cap \mathbb{T}, \\ \phi\big(\tau_1(x), \tau_2(y)\big) \leqslant a^{\frac{1}{p}}(x, y) & \text{if } \tau_1(x) \leqslant x_0 \text{ or } \tau_2(y) \leqslant y_0,\ \forall (x, y) \in \widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0, \end{cases} \tag{3.30}$$

then

$$u(x, y) \leqslant \left\{G^{-1}\left[G\big(a(x, y)\big) + \int_{x_0}^{x}\int_{y_0}^{y} f(x, s, y, t)\Delta t\Delta s\right]\right\}^{\frac{1}{p}}, \quad (x, y) \in \widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0, \tag{3.31}$$

where G is defined the same as in (3.4).

Proof. Let $X \in \widehat{\mathbb{T}}_0$, $Y \in \widetilde{\mathbb{T}}_0$ be two fixed numbers, and $x \in [x_0, X] \cap \mathbb{T}$, $y \in [y_0, Y] \cap \mathbb{T}$. Denote

$$v(x, y) = a(X, Y) + \int_{x_0}^{x}\int_{y_0}^{y} f(x, s, y, t)\omega\big(u\big(\tau_1(s), \tau_2(t)\big)\big)\Delta t\Delta s.$$

Then we have

$$u(x, y) \leqslant v^{\frac{1}{p}}(x, y), \quad x \in [x_0, X] \cap \mathbb{T},\ y \in [y_0, Y] \cap \mathbb{T}. \tag{3.32}$$

Furthermore, for $x \in [x_0, X] \cap \mathbb{T}$, $y \in [y_0, Y] \cap \mathbb{T}$, if $\tau_1(x) \geqslant x_0$ and $\tau_2(y) \geqslant y_0$, then $\tau_1(x) \in [x_0, X] \cap \mathbb{T}$, $\tau_2(y) \in [y_0, Y] \cap \mathbb{T}$, and we have

$$u\big(\tau_1(x), \tau_2(y)\big) \leqslant v^{\frac{1}{p}}\big(\tau_1(x), \tau_2(y)\big) \leqslant v^{\frac{1}{p}}(x, y). \tag{3.33}$$

If $\tau_1(x) \leqslant x_0$ or $\tau_2(y) \leqslant y_0$, then from (3.29) we have

$$u\big(\tau_1(x), \tau_2(y)\big) = \phi\big(\tau_1(x), \tau_2(y)\big) \leqslant a^{\frac{1}{p}}(x, y) \leqslant a^{\frac{1}{p}}(X, Y) \leqslant v^{\frac{1}{p}}(x, y). \tag{3.34}$$

From (3.33) and (3.34) we always have

$$u\big(\tau_1(x), \tau_2(y)\big) \leqslant v^{\frac{1}{p}}(x, y), \quad x \in [x_0, X] \cap \mathbb{T},\ y \in [y_0, Y] \cap \mathbb{T}. \tag{3.35}$$

Collecting the information above we obtain

$$v(x, y) \leqslant a(X, Y) + \int_{x_0}^{x}\int_{y_0}^{y} f(x, s, y, t)\omega\big(v^{\frac{1}{p}}(s, t)\big)\Delta t\Delta s. \tag{3.36}$$

Let the right side of (3.36) be $z(x, y)$, then

$$v(x, y) \leqslant z(x, y), \quad x \in [x_0, X] \cap \mathbb{T},\ y \in [y_0, Y] \cap \mathbb{T}. \tag{3.37}$$

Considering $[\int_{y_0}^y f(x, s, y, t)\Delta t]_x^\Delta \geqslant 0$, by Lemma 3.1 we have

$$
\begin{aligned}
z_x^{\Delta}(x,y) &= \int_{x_0}^{x}\left[\int_{y_0}^{y} f(x,s,y,t)\omega\big(v^{\frac{1}{p}}(s,t)\big)\Delta t\right]_x^{\Delta}\Delta s + \int_{y_0}^{y} f\big(\sigma(x),x,y,t\big)\omega\big(v^{\frac{1}{p}}(x,t)\big)\Delta t \\
&\leqslant \left\{\int_{x_0}^{x}\left[\int_{y_0}^{y} f(x,s,y,t)\Delta t\right]_x^{\Delta}\Delta s + \int_{y_0}^{y} f\big(\sigma(x),x,y,t\big)\Delta t\right\}\omega\big(z^{\frac{1}{p}}(x,y)\big) \\
&= \left[\int_{x_0}^{x}\int_{y_0}^{y} f(x,s,y,t)\Delta t\Delta s\right]_x^{\Delta}\omega\big(z^{\frac{1}{p}}(x,y)\big),
\end{aligned}
$$

where $z_x^{\Delta}(x,y)$ denotes the *partial delta derivative* of $z(x,y)$ with respect to x.

Furthermore,

$$
\frac{z_x^{\Delta}(x,y)}{\omega(z^{\frac{1}{p}}(x,y))} \leqslant \left[\int_{x_0}^{x}\int_{y_0}^{y} f(x,s,y,t)\Delta t\Delta s\right]_x^{\Delta}. \tag{3.38}
$$

An integration for (3.38) with respect to x from x_0 to x yields

$$
G\big(z(x,y)\big) - G\big(z(x_0,y)\big) \leqslant \int_{x_0}^{x}\int_{y_0}^{y} f(x,s,y,t)\Delta t\Delta s,
$$

which is followed by

$$
\begin{aligned}
z(x,y) &\leqslant G^{-1}\left[G\big(z(x_0,y)\big) + \int_{x_0}^{x}\int_{y_0}^{y} f(x,s,y,t)\Delta t\Delta s\right] \\
&= G^{-1}\left[G\big(a(X,Y)\big) + \int_{x_0}^{x}\int_{y_0}^{y} f(x,s,y,t)\Delta t\Delta s\right], \quad x \in [x_0,X]\cap\mathbb{T},\ y \in [y_0,Y]\cap\mathbb{T}.
\end{aligned} \tag{3.39}
$$

Combining (3.32), (3.37) and (3.39) we obtain

$$
u(x,y) \leqslant \left\{G^{-1}\left[G\big(a(X,Y)\big) + \int_{x_0}^{x}\int_{y_0}^{y} f(x,s,y,t)\Delta t\Delta s\right]\right\}^{\frac{1}{p}}, \quad x \in [x_0,X]\cap\mathbb{T},\ y \in [y_0,Y]\cap\mathbb{T}. \tag{3.40}
$$

Take $x = X$, $y = Y$ in (3.40), and considering $X \in \widehat{\mathbb{T}}_0$, $Y \in \widetilde{\mathbb{T}}_0$ are arbitrary, then after substituting X, Y with x, y we obtain the desired inequality. □

Remark 3.4. Theorem 3.4 unifies some known continuous and discrete analysis in the literature. If we take $\mathbb{T} = \mathbb{R}$, $p = 1$, $x_0 = y_0 = 0$ or $\mathbb{T} = \mathbb{R}$, $p = 1$, $x_0 = y_0 = 0$, $f(x,s,y,t) = f_1(x,y)f_2(s,t)$, then Theorem 3.4 reduces to [4, Theorem 2.3] and [4, Theorem 2.4] respectively. If we take $\mathbb{T} = \mathbb{Z}$, $x_0 = y_0 = 0$, $\tau_1(x) = x$, $\tau_2(y) = y$, $b(x,y) \equiv 1$, $a(x,y) \equiv C$, $f(x,s,y,t) = f_1(x,y)f_2(s,t)$, then Theorem 3.4 reduces to [5, Theorem 2.1].

Theorem 3.5. *Suppose $u, a, b, f_i, g_i, h_i \in C_{rd}(\widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R}_+)$, $i = 1,2,\ldots,n$, and a, b are nondecreasing. $\eta \in C(\mathbb{R}_+,\mathbb{R}_+)$ is strictly increasing. ω is the same as in Theorem* 3.1. *$\tau_{1i} \in (\widehat{\mathbb{T}}_0, \mathbb{T})$, $\tau_{1i}(x) \leqslant x$, $i = 1,2,\ldots,n$. $-\infty < \alpha = \inf\{\min\{\tau_{1i}(x),\ i = 1,2,\ldots,n\},\ x \in \widehat{\mathbb{T}}_0\} \leqslant x_0$. $\tau_{2i} \in (\widetilde{\mathbb{T}}_0, \mathbb{T})$, $\tau_{2i}(y) \leqslant y$, $i = 1,2,\ldots,n$. $-\infty < \beta = \inf\{\min\{\tau_{2i}(y),\ i = 1,2,\ldots,n\},\ y \in \widetilde{\mathbb{T}}_0\} \leqslant y_0$. $\phi \in C_{rd}(([\alpha,x_0]\times[\beta,y_0])\cap\mathbb{T}^2,\mathbb{R}_+)$. $q > 0$ is a constant. If for $(x,y) \in \widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0$, $u(x,y)$ satisfies the following inequality*

$$
\begin{aligned}
\eta\big(u(x,y)\big) \leqslant a(x,y) + b(x,y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y}\Bigg[& f_i(s,t)u^q\big(\tau_{1i}(s),\tau_{2i}(t)\big)\omega\big(u\big(\tau_{1i}(s),\tau_{2i}(t)\big)\big) \\
&+ g_i(s,t)u^q\big(\tau_{1i}(s),\tau_{2i}(t)\big) + \int_{x_0}^{s}\int_{y_0}^{t} h_i(\xi,\zeta)u^q\big(\tau_{1i}(\xi),\tau_{2i}(\zeta)\big)\Delta\zeta\Delta\xi\Bigg]\Delta t\Delta s,
\end{aligned} \tag{3.41}
$$

with the initial condition

$$\begin{cases} \eta\big(u(x,y)\big)=\phi(x,y) & \text{if } x\in[\alpha,x_0]\cap\mathbb{T} \text{ or } y\in[\beta,y_0]\cap\mathbb{T}, \\ \phi\big(\tau_{1i}(x),\tau_{2i}(y)\big)\leqslant a(x,y) & \text{if } \tau_{1i}(x)\leqslant x_0 \text{ or } \tau_{2i}(y)\leqslant y_0,\ i=1,2,\ldots,n,\ \forall(x,y)\in\widehat{\mathbb{T}}_0\times\widetilde{\mathbb{T}}_0, \end{cases} \tag{3.42}$$

then

$$u(x,y)\leqslant\eta^{-1}\Bigg\{\widehat{G}^{-1}\Bigg\{\widehat{H}^{-1}\Bigg\{\widehat{H}\Bigg\{\widehat{G}\big(a(x,y)\big)+b(x,y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y}\Bigg[g_i(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}h_i(\xi,\zeta)\Delta\zeta\,\Delta\xi\Bigg]\Delta t\,\Delta s\Bigg\}$$

$$+b(x,y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y}f_i(s,t)\Delta t\,\Delta s\Bigg\}\Bigg\}\Bigg\},\quad (x,y)\in\widehat{\mathbb{T}}_0\times\widetilde{\mathbb{T}}_0, \tag{3.43}$$

where $\widehat{G}$, $\widehat{H}$ are two increasing bijective functions, and

$$\begin{cases} \big[\widehat{G}\big(v(x,y)\big)\big]_x^{\Delta}=\dfrac{v_x^{\Delta}(x,y)}{(\eta^{-1}(v(x,y)))^q}, \\ \big[\widehat{H}\big(z(x,y)\big)\big]_x^{\Delta}=\dfrac{z_x^{\Delta}(x,y)}{\omega(\eta^{-1}(\widehat{G}^{-1}(z(x,y))))}. \end{cases} \tag{3.44}$$

Proof. Let $X\in\widehat{\mathbb{T}}_0$, $Y\in\widetilde{\mathbb{T}}_0$ be two fixed numbers, and $x\in[x_0,X]\cap\mathbb{T}$, $y\in[y_0,Y]\cap\mathbb{T}$. Denote

$$v(x,y)=a(X,Y)+b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y}\Bigg[f_i(s,t)u^q\big(\tau_{1i}(s),\tau_{2i}(t)\big)\omega\big(u\big(\tau_{1i}(s),\tau_{2i}(t)\big)\big)$$

$$+g_i(s,t)u^q\big(\tau_{1i}(s),\tau_{2i}(t)\big)+\int_{x_0}^{s}\int_{y_0}^{t}h_i(\xi,\zeta)u^q\big(\tau_{1i}(\xi),\tau_{2i}(\zeta)\big)\Delta\zeta\,\Delta\xi\Bigg]\Delta t\,\Delta s. \tag{3.45}$$

Then we have

$$u(x,y)\leqslant\eta^{-1}\big(v(x,y)\big),\quad x\in[x_0,X]\cap\mathbb{T},\ y\in[y_0,Y]\cap\mathbb{T}. \tag{3.46}$$

Similar to the process of (3.33)–(3.35) we have

$$u\big(\tau_{1i}(x),\tau_{2i}(y)\big)\leqslant\eta^{-1}\big(v(x,y)\big),\quad x\in[x_0,X]\cap\mathbb{T},\ y\in[y_0,Y]\cap\mathbb{T},\ i=1,2,\ldots,n. \tag{3.47}$$

Furthermore,

$$v_x^{\Delta}(x,y)=b(X,Y)\sum_{i=1}^{n}\int_{y_0}^{y}\Bigg[f_i(x,t)u^q\big(\tau_{1i}(x),\tau_{2i}(t)\big)\omega\big(u\big(\tau_{1i}(x),\tau_{2i}(t)\big)\big)$$

$$+g_i(x,t)u^q\big(\tau_{1i}(x),\tau_{2i}(t)\big)+\int_{x_0}^{x}\int_{y_0}^{t}h_i(\xi,\zeta)u^q\big(\tau_{1i}(\xi),\tau_{2i}(\zeta)\big)\Delta\zeta\,\Delta\xi\Bigg]\Delta t$$

$$\leqslant\Bigg\{b(X,Y)\sum_{i=1}^{n}\int_{y_0}^{y}\Bigg[f_i(x,t)\omega\big(\eta^{-1}\big(v(x,t)\big)\big)+g_i(x,t)+\int_{x_0}^{x}\int_{y_0}^{t}h_i(\xi,\zeta)\Delta\zeta\,\Delta\xi\Bigg]\Delta t\Bigg\}\big(\eta^{-1}\big(v(x,y)\big)\big)^q, \tag{3.48}$$

which implies

$$\frac{v_x^{\Delta}(x,y)}{(\eta^{-1}(v(x,y)))^q}\leqslant b(X,Y)\sum_{i=1}^{n}\int_{y_0}^{y}\Bigg[f_i(x,t)\omega\big(\eta^{-1}\big(v(x,t)\big)\big)+g_i(x,t)+\int_{x_0}^{x}\int_{y_0}^{t}h_i(\xi,\zeta)\Delta\zeta\,\Delta\xi\Bigg]\Delta t. \tag{3.49}$$

An integration for (3.49) with respect to x from x_0 to x yields

$$\widehat{G}\big(v(x,y)\big)-\widehat{G}\big(v(x_0,y)\big)\leqslant b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y}\Bigg[f_i(s,t)\omega\big(\eta^{-1}\big(v(s,t)\big)\big)+g_i(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}h_i(\xi,\zeta)\Delta\zeta\,\Delta\xi\Bigg]\Delta t\,\Delta s,$$

which is followed by

$$v(x,y) \leqslant \widehat{G}^{-1}\left\{\widehat{G}\big(a(X,Y)\big) + b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y}\left[f_i(s,t)\omega\big(\eta^{-1}\big(v(s,t)\big)\big) + g_i(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} h_i(\xi,\zeta)\Delta\zeta\Delta\xi\right]\Delta t\Delta s\right\}.$$

Let

$$z(x,y) = \widehat{G}\big(a(X,Y)\big) + b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{X}\int_{y_0}^{Y}\left[g_i(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} h_i(\xi,\zeta)\Delta\zeta\Delta\xi\right]\Delta t\Delta s$$
$$+ b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)\omega\big(\eta^{-1}\big(v(s,t)\big)\big)\Delta t\Delta s,$$

then we have

$$v(x,y) \leqslant \widehat{G}^{-1}\big(z(x,y)\big), \quad x\in[x_0,X]\cap\mathbb{T},\ y\in[y_0,Y]\cap\mathbb{T}, \tag{3.50}$$

and furthermore

$$z_x^{\Delta}(x,y) = b(X,Y)\sum_{i=1}^{n}\int_{y_0}^{y} f_i(x,t)\omega\big(\eta^{-1}\big(v(x,t)\big)\big)\Delta t \leqslant \left[b(X,Y)\sum_{i=1}^{n}\int_{y_0}^{y} f_i(x,t)\Delta t\right]\omega\big(\eta^{-1}\big(\widehat{G}^{-1}\big(z(x,y)\big)\big)\big),$$

which is followed by

$$\frac{z_x^{\Delta}(x,y)}{\omega(\eta^{-1}(\widehat{G}^{-1}(z(x,y))))} \leqslant b(X,Y)\sum_{i=1}^{n}\int_{y_0}^{y} f_i(x,t)\Delta t. \tag{3.51}$$

An integration for (3.51) with respect to x from x_0 to x yields

$$\widehat{H}\big(z(x,y)\big) - \widehat{H}\big(z(x_0,y)\big) \leqslant b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)\Delta t\Delta s,$$

that is,

$$z(x,y) \leqslant \widehat{H}^{-1}\left\{\widehat{H}\big[z(x_0,y)\big] + b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)\Delta t\Delta s\right\}$$
$$= \widehat{H}^{-1}\left\{\widehat{H}\left\{\widehat{G}\big(a(X,Y)\big) + b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{X}\int_{y_0}^{Y}\left[g_i(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} h_i(\xi,\zeta)\Delta\zeta\Delta\xi\right]\Delta t\Delta s\right\}\right.$$
$$\left. + b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)\Delta t\Delta s\right\}, \quad x\in[x_0,X]\cap\mathbb{T},\ y\in[y_0,Y]\cap\mathbb{T}. \tag{3.52}$$

Combining (3.46), (3.50) and (3.52) we obtain

$$u(x,y) \leqslant \eta^{-1}\left\{\widehat{G}^{-1}\left\{\widehat{H}^{-1}\left\{\widehat{H}\left\{\widehat{G}\big(a(X,Y)\big) + b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{X}\int_{y_0}^{Y}\left[g_i(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} h_i(\xi,\zeta)\Delta\zeta\Delta\xi\right]\Delta t\Delta s\right\}\right.\right.\right.$$
$$\left.\left.\left. + b(X,Y)\sum_{i=1}^{n}\int_{x_0}^{x}\int_{y_0}^{y} f_i(s,t)\Delta t\Delta s\right\}\right\}\right\}, \quad x\in[x_0,X]\cap\mathbb{T},\ y\in[y_0,Y]\cap\mathbb{T}. \tag{3.53}$$

Take $x = X$, $y = Y$ in (3.53), and considering $X\in\widehat{\mathbb{T}}_0$, $Y\in\widetilde{\mathbb{T}}_0$ are arbitrary, then after substituting X, Y with x, y we obtain the desired inequality. □

Remark 3.5. If we take $\mathbb{T} = \mathbb{R}$, $h_i(x,y) \equiv 0$, then Theorem 3.5 reduces to [6, Theorem 2.1].

4. Applications

In this section, we will present some simple applications for our results, and will try to show they are useful in deriving bounds of solutions of certain dynamic equations.

Example 1. Consider the following delay differential equations on time scales

$$u^{\Delta}(t)=F\left(t,u\bigl(\tau(t)\bigr),\int_{t_0}^{t}M\bigl(\xi,u\bigl(\tau(\xi)\bigr)\bigr)\Delta\xi\right),\tag{4.1}$$

with the initial condition

$$\begin{cases}u(t_0)=C,\\ u(t)=\phi(t)\quad \text{if } t\in[\alpha,t_0]\cap\mathbb{T},\\ \phi\bigl(\tau(t)\bigr)\leqslant|C|^{\frac{1}{p}},\quad \forall t\in\mathbb{T}_0 \text{ if } \tau(t)\leqslant t_0,\end{cases}\tag{4.2}$$

where $u\in C_{rd}(\mathbb{T}_0,\mathbb{R}\backslash\{0\})$, C is a constant with $C\neq 0$, τ, α are the same as in Theorem 3.1, $\phi\in C_{rd}([\alpha,t_0]\cap\mathbb{T},\mathbb{R})$.

Lemma 4.1. *(See [23, Theorem 1.93].) Assume that $v:\mathbb{T}\to\mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}}:=v(\mathbb{T})$ is a time scale. Let $\omega:\widetilde{\mathbb{T}}\to\mathbb{R}$. If $v^{\Delta}(t)$ and $\omega^{\widetilde{\Delta}}(v(t))$ exist on $\mathbb{T}$, then*

$$(\omega\circ v)^{\Delta}=\bigl(\omega^{\widetilde{\Delta}}\circ v\bigr)v^{\Delta}.$$

Theorem 4.1. *Suppose $u(t)$ is a solution of (4.1)–(4.2), and assume $|F(t,u,v)|\leqslant f(t)\psi(|u|)\omega(|u|)+g(t)\psi|u|+|v|$, $|M(t,u)|\leqslant h(t)\psi|u|$, where $f,g,h\in C_{rd}(\mathbb{T}_0,\mathbb{R}_+)$ with $f(t)$ not equivalent to zero, $\omega(u)=\sqrt{\overline{\overline{\sigma}}(\sqrt{u})}+u^{\frac{1}{4}}$, $\psi(u)=\sqrt{\overline{\sigma}(u)}+\sqrt{u}$, $\forall u\in\mathbb{R}_+$, and $\overline{\sigma}$, $\overline{\overline{\sigma}}$ are defined as below. Furthermore, let $\widetilde{G}$, $\widetilde{H}$ are functions defined on $\mathbb{R}_+$, and $\widetilde{G}(u)=\sqrt{u}$, $\widetilde{H}(u)=\sqrt{u}$, $\forall u\in\mathbb{R}_+$, then we have*

$$\bigl|u(t)\bigr|\leqslant\left[\sqrt{\sqrt{|C|}+\int_{t_0}^{t}\left[g(s)+\int_{t_0}^{s}h(\xi)\Delta\xi\right]\Delta s}+\int_{t_0}^{t}f(s)\Delta s\right]^4,\quad t\in\mathbb{T}_0.$$

Proof. Denote $\overline{\mathbb{T}}_0:=v(\mathbb{T}_0)$, $\overline{\overline{\mathbb{T}}}_0:=z(\mathbb{T}_0)$, where $t\in\mathbb{T}_0$, and $v(t)$, $z(t)$ are defined as in Theorem 3.3 (with $a(t)$, $f(t,s)$, $g(t,s)$, u replaced by $|C|$, $f(s)$, $g(s)$, $|u|$ respectively). Let $\overline{\Delta}$ and $\overline{\overline{\Delta}}$ denote the *delta derivative* on $\overline{\mathbb{T}}_0$ and $\overline{\overline{\mathbb{T}}}_0$ respectively, and $\overline{\sigma}$ and $\overline{\overline{\sigma}}$ denote the forward jump operators on $\overline{\mathbb{T}}_0$ and $\overline{\overline{\mathbb{T}}}_0$ respectively. By a close look at the definition of $v(t)$, $z(t)$ in Theorem 3.3 we can see they are both strictly increasing. Then by Lemma 4.1 we obtain

$$\begin{cases}\bigl[\widetilde{G}\bigl(v(t)\bigr)\bigr]^{\Delta}=\widetilde{G}^{\overline{\Delta}}(v)v^{\Delta}(t)=\dfrac{v^{\Delta}(t)}{\sqrt{\overline{\sigma}(v(t))}+\sqrt{v(t)}}=\dfrac{v^{\Delta}(t)}{\psi(v(t))}=\dfrac{v^{\Delta}(t)}{\psi(\eta^{-1}(v(t)))},\\ \bigl[\widetilde{H}\bigl(z(t)\bigr)\bigr]^{\Delta}=\widetilde{H}^{\overline{\overline{\Delta}}}(z)z^{\Delta}(t)=\dfrac{z^{\Delta}(t)}{\sqrt{\overline{\overline{\sigma}}(z(t))+\sqrt{z(t)}}}=\dfrac{z^{\Delta}(t)}{\omega(z^2(t))}=\dfrac{z^{\Delta}(t)}{\omega(\widetilde{G}^{-1}(z(t)))}=\dfrac{z^{\Delta}(t)}{\omega(\eta^{-1}(\widetilde{G}^{-1}(z(t))))}.\end{cases}$$

Considering $u(t_0)=C$, the equivalent integral equation of Eq. (4.1) is denoted by

$$u(t)=C+\int_{t_0}^{t}F\left(t,u\bigl(\tau(t)\bigr),\int_{t_0}^{t}M\bigl(\xi,u\bigl(\tau(\xi)\bigr)\bigr)\Delta\xi\right).\tag{4.3}$$

Then under the conditions of Theorem 4.1, we have

$$\begin{aligned}\bigl|u(t)\bigr|&\leqslant|C|+\left|\int_{t_0}^{t}F\left(s,u\bigl(\tau(s)\bigr),\int_{t_0}^{s}M\bigl(\xi,u\bigl(\tau(\xi)\bigr)\bigr)\Delta\xi\right)\Delta s\right|\\ &\leqslant|C|+\int_{t_0}^{t}\left|F\left(s,u\bigl(\tau(s)\bigr),\int_{t_0}^{s}M\bigl(\xi,u\bigl(\tau(\xi)\bigr)\bigr)\Delta\xi\right)\right|\Delta s\end{aligned}$$

$$\leqslant |C| + \int_{t_0}^{t} \left[f(s)\psi\big(|u|\big)\omega\big(\big|u\big(\tau(s)\big)\big|\big) + g(s)\psi\big|u\big(\tau(s)\big)\big| + \left| \int_{t_0}^{s} M\big(\xi, u\big(\tau(\xi)\big)\big)\Delta\xi \right| \right] \Delta s$$

$$\leqslant |C| + \int_{t_0}^{t} \left[f(s)\psi\big(|u|\big)\omega\big(\big|u\big(\tau(s)\big)\big|\big) + g(s)\psi\big|u\big(\tau(s)\big)\big| + \int_{t_0}^{s} h(\xi)\psi\big|u\big(\tau(\xi)\big)\big|\Delta\xi \right] \Delta s. \tag{4.4}$$

Then a suitable application of Theorem 3.3 to (4.4) yields

$$\big|u(t)\big| \leqslant \widetilde{G}^{-1}\left\{ \widetilde{H}^{-1}\left\{ \widetilde{H}\left\{ \widetilde{G}\big(|C|\big) + \int_{t_0}^{t}\left[g(s) + \int_{t_0}^{s} h(\xi)\Delta\xi \right]\Delta s \right\} + \int_{t_0}^{t} f(s)\Delta s \right\}\right\}, \quad t \in \mathbb{T}_0.$$

Using the definition of $\widetilde{G}$, $\widetilde{H}$, we simplify the inequality above as follows

$$\big|u(t)\big| \leqslant \left[\sqrt{\sqrt{|C|} + \int_{t_0}^{t}\left[g(s) + \int_{t_0}^{s} h(\xi)\Delta\xi \right]\Delta s} + \int_{t_0}^{t} f(s)\Delta s \right]^4, \quad t \in \mathbb{T}_0,$$

which is the desired result. $\square$

Example 2. Consider the following integral equation on time scales

$$u(x,y) = a(x) + b(y) + \int_{x_0}^{x}\int_{y_0}^{y} F\big(x,s,y,t,u\big(\tau_1(s),\tau_2(t)\big)\big)\Delta t\,\Delta s, \tag{4.5}$$

with the initial condition

$$\begin{cases} u(x_0,y) = b(y), \qquad u(x,y_0) = a(x), \\ a(x_0) = b(y_0) = 0, \\ u(x,y) = \phi(x,y) \quad \text{if } x \in [\alpha, x_0] \cap \mathbb{T} \text{ or } y \in [\beta, y_0] \cap \mathbb{T}, \\ \phi\big(\tau_1(x),\tau_2(y)\big) \leqslant \big|a(x)+b(y)\big|, \quad \forall (x,y) \in \widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0 \text{ if } \tau_1(x) \leqslant x_0 \text{ or } \tau_2(y) \leqslant y_0, \end{cases} \tag{4.6}$$

where $u \in C_{rd}(\widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R})$, and τ_1, τ_2, α, β are the same as in Theorem 3.4, $\phi \in C_{rd}(([\alpha, x_0] \times [\beta, y_0]) \cap \mathbb{T}^2, \mathbb{R})$.

Theorem 4.2. *Suppose $u(x,y)$ is a solution of (4.5)–(4.6), and assume $|F(x,s,y,t,u)| \leqslant f(x,s,y,t)\omega(|u|)$, $|a(x)+b(y)| \leqslant k(x,y)$, where $k \in C_{rd}(\widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0, \mathbb{R}_+)$, $f \in C_{rd}(\widehat{\mathbb{T}}_0^2 \times \widetilde{\mathbb{T}}_0^2, \mathbb{R}_+)$ with k, f not equivalent to zero, and k is nondecreasing, $\omega(u) = \sqrt[3]{(\widehat{\sigma}(u))^2} + \sqrt[3]{\widehat{\sigma}(u)u} + \sqrt[3]{u^2}$, $\forall u \in \mathbb{R}_+$, and $\widehat{\sigma}$ is defined as below. Furthermore, let G be a function defined on $\mathbb{R}_+$, and $G(u) = \sqrt[3]{u}$, $\forall u \in \mathbb{R}_+$, then the following inequality holds*

$$\big|u(x,y)\big| \leqslant \left[\sqrt[3]{k(x,y)} + \int_{x_0}^{x}\int_{y_0}^{y} f(x,s,y,t)\Delta t\,\Delta s \right]^3, \quad (x,y) \in \widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0. \tag{4.7}$$

Proof. Denote $\widehat{\widehat{\mathbb{T}}}_0 := z(\widehat{\mathbb{T}}_0, y)$, where $x \in \widehat{\mathbb{T}}_0$, and $z(x,y)$ is defined as in Theorem 3.4 (with $a(x,y)$, u replaced by $k(x,y)$, $|u|$ respectively, and $p = 1$). Let $\widehat{\Delta}$ denote the *delta derivative* on $\widehat{\widehat{\mathbb{T}}}_0$, and $\widehat{\sigma}$ denote the forward jump operator on $\widehat{\widehat{\mathbb{T}}}_0$. By a close look at the definition of $z(x,y)$ in Theorem 3.4 we can see it is strictly increasing. Then by Lemma 4.1 we obtain

$$\big[G\big(z(x,y)\big)\big]_x^{\Delta} = G^{\widehat{\Delta}}(z) z_x^{\Delta}(x,y) = \frac{v_x^{\Delta}(x,y)}{\sqrt[3]{(\widehat{\sigma}(z))^2} + \sqrt[3]{\widehat{\sigma}(z)z} + \sqrt[3]{z^2}} = \frac{z_x^{\Delta}(x,y)}{\omega(z(x,y))}.$$

From (4.5) we have

$$\big|u(x,y)\big| \leqslant \big|a(x)+b(y)\big| + \int_{x_0}^{x}\int_{y_0}^{y} \big|F\big(x,s,y,t,u\big(\tau_1(s),\tau_2(t)\big)\big)\big|\Delta t\,\Delta s$$

$$\leqslant k(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} f(x,s,y,t)\omega\big(\big|u\big(\tau_1(s),\tau_2(t)\big)\big|\big)\Delta t\,\Delta s. \tag{4.8}$$

Then a suitable application of Theorem 3.4 (with $p = 1$) to (4.8) yields

$$\left|u(x, y)\right| \leqslant G^{-1}\left[G\big(k(x, y)\big) + \int_{x_0}^{x}\int_{y_0}^{y} f(x, s, y, t)\Delta t\Delta s\right], \quad (x, y) \in \widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0.$$

Using the definition of G, we simplify the inequality above as follows

$$\left|u(x, y)\right| \leqslant \left[\sqrt[3]{k(x, y)} + \int_{x_0}^{x}\int_{y_0}^{y} f(x, s, y, t)\Delta t\Delta s\right]^3, \quad (x, y) \in \widehat{\mathbb{T}}_0 \times \widetilde{\mathbb{T}}_0,$$

which is the desired result. □

5. Conclusions

In this paper, we establish some new Gronwall–Bellman type nonlinear delay integral inequalities on time scales, which extend many known inequalities in the literature, and provide a handy tool for deriving bounds of solutions of delay dynamic equations on time scales. Since our results are established on arbitrary time scales, as a result, our Theorems 3.2 and 3.4 unify some known continuous and discrete analysis in the literature as stated in Remarks 3.2 and 3.4. Inequalities of other types on time scales can be supposed to further research.

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